

Graph states in phase space

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Abstract. The phase space for a system of n qubits is a discrete grid of $2^n \times 2^n$ points, whose axes are labeled in terms of the elements of the finite field \mathbb{F}_{2^n} to endow it with proper geometrical properties. We analyze the representation of graph states in that phase space, showing that these states can be identified with a class of nonsingular curves. We provide an algebraic representation of the most relevant quantum operations acting on these states and discuss the advantages of this approach.

PACS numbers: 03.65.Aa, 03.65.Ta, 03.65.Ud, 03.67.Mn

1. Introduction

Graph states represent a versatile class of entangled states of uttermost importance in quantum information. Originally proposed for implementing measurement-based quantum computation [2, 3, 4, 5], they have found numerous applications in other problems such as quantum error correction [6], secure quantum communication [7], entanglement purification [8], and to demonstrate fractional braiding statistics of anyons [9]. Furthermore, instances of this family, such as Greenberger-Horne-Zeilinger (GHZ) states and cluster states, play an essential role in fundamental tests of quantum nonlocality [10, 11]. Consequently, a great deal of effort has been devoted to theoretically understand their properties [12, 13] and to create and manipulate them experimentally [14, 15, 16, 17].

In addition to all these applications, the graph-state formalism is a useful abstraction that permits, in principle, a detailed (although not exhaustive) classification of n -qubit entanglement [18, 19, 20]. However, the number of graphs grows very fast with n , so such a job becomes rather involved for many qubits. Moreover, for these high-dimensional systems, the action of nonlocal operations is difficult to interpret.

An important remark in this respect is that other quantum tasks (such as, e.g., tomography) can be most efficiently implemented in phase space. For n qubits, the phase space is a discrete grid of $2^n \times 2^n$ points [21, 22, 23, 24, 25], whose axes are labelled by the discrete Galois field \mathbb{F}_{2^n} in order to preserve some intuitive geometrical properties [26]. In this way, a *bona fide* Wigner function can be introduced [27, 28, 29, 30, 31]. One natural question that comes to mind is whether this phase-space picture may be of any help in reinterpreting complex issues of graph states. Our work provides a positive answer: graph states turn out to be nothing but discrete curves in phase space [32], with properties closely related to other intriguing notions such as unbiasedness [33]. In this way, quantum operations appear as transformations on these curves, with very simple and elegant properties.

2. Curves in phase space

A qubit is realized as a state in a two-dimensional Hilbert space. It is customary to choose two normalized orthogonal states, $\{|0\rangle, |1\rangle\}$, to serve as a computational basis. The unitary operators

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|, \quad \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad (2.1)$$

generate the Pauli group \mathcal{P}_1 of a single qubit under matrix multiplication [34]. The elements of this group are known as Pauli operators and provide a basis of unitary operators on the Hilbert space.

For n qubits, a compact way of labeling both states and elements of the corresponding Pauli group \mathcal{P}_n consists in using the finite field \mathbb{F}_{2^n} (the reader interested in more mathematical details is referred, to the excellent monograph by Lidl and Niederreiter [26]). This can be considered as a linear space spanned by an abstract basis $\{\theta_1, \dots, \theta_n\}$, so that given a field element α (henceforth, field elements will be denoted by Greek letters) the expansion

$$\alpha = \sum_{i=1}^n a_i \theta_i, \quad a_i \in \mathbb{Z}_2, \quad (2.2)$$

allows us the identification $\alpha \Leftrightarrow (a_1, \dots, a_n)$. Moreover, the basis can be chosen to be orthonormal with respect to the trace operation (the self-dual basis); that is,

$$\text{tr}(\theta_i \theta_j) = \delta_{ij}, \quad (2.3)$$

where $\text{tr}(\alpha) = \alpha + \alpha^2 + \dots + \alpha^{2^{n-1}}$, which actually maps $\mathbb{F}_{2^n} \mapsto \mathbb{Z}_2$. In this way, we associate each qubit with a particular element of the self-dual basis: qubit $i \Leftrightarrow \theta_i$.

We denote by $|\alpha\rangle$, with $\alpha \in \mathbb{F}_{2^n}$, an orthonormal basis in the Hilbert space of the system. Operationally, the elements of the basis can be labeled by powers of a primitive element (i.e., a root of a minimal irreducible polynomial, called the primitive polynomial), and reads $\{|0\rangle, |\sigma\rangle, \dots, |\sigma^{2^n-1} = 1\rangle\}$. These vectors are eigenvectors of the operators Z_β belonging to the generalized Pauli group \mathcal{P}_n , whose generators are

$$Z_\lambda = \sum_{\alpha} \chi(\lambda \alpha) |\alpha\rangle\langle \alpha|, \quad X_\lambda = \sum_{\alpha} |\alpha + \lambda\rangle\langle \alpha|. \quad (2.4)$$

Here the additive characters χ are defined as $\chi(\alpha) = \exp[i\pi \text{tr}(\alpha)]$. We have then

$$Z_\alpha X_\beta = \chi(\alpha\beta) X_\beta Z_\alpha, \quad (2.5)$$

which is the discrete counterpart of the Heisenberg-Weyl algebra for continuous variables.

The operators (2.4) can be factorized into tensor products of powers of single-particle Pauli operators. This factorization can be carried out by mapping each element of \mathbb{F}_{2^n} onto an ordered set of natural numbers according to

$$Z_\alpha = \sigma_z^{a_1} \otimes \dots \otimes \sigma_z^{a_n}, \quad X_\beta = \sigma_x^{b_1} \otimes \dots \otimes \sigma_x^{b_n}, \quad (2.6)$$

where $a_i = \text{tr}(\alpha \theta_i)$ and $b_i = \text{tr}(\beta \theta_i)$ are the corresponding expansion coefficients for α and β in the self-dual basis.

The commutator of two monomials in \mathcal{P}_n is

$$[Z_{\alpha_1} X_{\beta_1}, Z_{\alpha_2} X_{\beta_2}] = [\chi(\alpha_1 \beta_2 + \beta_1 \alpha_2) - 1] Z_{\alpha_2} X_{\beta_2} Z_{\alpha_1} X_{\beta_1}, \quad (2.7)$$

so they commute when

$$\text{tr}(\alpha_1 \beta_2 + \beta_1 \alpha_2) = 0. \quad (2.8)$$

We next recall [21, 22, 23] that the grid defining the phase space for n qubits can be appropriately labeled by the discrete points (α, β) , which are precisely the indices of the operators Z_α and X_β : α is the “horizontal” axis and β the “vertical” one.

From this perspective it seems reasonable to consider a (parametric) curve in this discrete grid as a set of $2^n + 1$ points $(\alpha(\kappa), \beta(\kappa))$, where the parameter κ runs through the field \mathbb{F}_{2^n} . The curve pass through the origin when $(\alpha(0), \beta(0)) = (0, 0)$, and it is nonsingular (i.e., with no self-intersection) if all the pairs $(\alpha(\kappa), \beta(\kappa))$ are different.

If we represent the curve as

$$\alpha(\kappa) = \sum_{i=0}^{n-1} \alpha_i \kappa^{2^i}, \quad \beta(\kappa) = \sum_{i=0}^{n-1} \beta_i \kappa^{2^i}, \quad (2.9)$$

with $\alpha_i, \beta_i \in \mathbb{F}_{2^n}$, we will say that the curve is additive commutative whenever

$$\sum_{i \neq j} \text{tr}(\alpha_i \beta_j) = 0. \quad (2.10)$$

Due to the form in equation (2.9), the following property is automatically fulfilled

$$\alpha(\kappa + \kappa') = \alpha(\kappa) + \alpha(\kappa'), \quad \beta(\kappa + \kappa') = \beta(\kappa) + \beta(\kappa'), \quad (2.11)$$

which means that by summing the coordinates of any two points of the curve we obtain another point on the curve. The condition (2.10) guarantees that the monomials labeled with points of such a curve form indeed an Abelian subgroup under multiplication [notice the condition (2.8)]; i. e.,

$$[Z_{\alpha(\kappa)} X_{\beta(\kappa)}, Z_{\alpha(\kappa')} X_{\beta(\kappa')}] = 0. \quad (2.12)$$

Observe also (2.11) implies that this group is isomorphous to $\mathbb{Z}_2^{\oplus n}$.

The 2^n monomials $\{Z_{\alpha(\kappa)} X_{\beta(\kappa)}\}$ constitute thus the whole set of stabilizer operators [34] and n of them appropriately chosen can be considered as generators. This means that one can pick n specific points that generate the whole curve by simple addition. The simplest way is to pick the horizontal coordinates as the points corresponding to the elements of the basis $\alpha = \{\theta_1, \dots, \theta_n\}$, the order of points being unessential.

The simplest form of additive commutative curves are the straight lines

$$\alpha(\kappa) = \mu \kappa, \quad \beta(\kappa) = \nu \kappa, \quad (2.13)$$

which can be represented in the regular form $\alpha = 0, \beta = \lambda \alpha$. It is a well established result [35, 36] that the operators $\{Z_\alpha X_{\beta=\lambda\alpha}\}$ associated to straight lines commute for any fixed value of $\lambda \in \mathbb{F}_{2^n}$, while the eigenstates of the set $\{Z_\alpha\}$ define the standard logical basis.

A curve is called nondegenerate (or regular) when can be represented in the explicit form

$$\beta = f(\alpha) = \sum_{i=0}^{n-1} \phi_i \alpha^{2^i} \quad \text{or} \quad \alpha = g(\beta) = \sum_{i=0}^{n-1} \psi_i \beta^{2^i}, \quad (2.14)$$

with $\phi_i, \psi_i \in \mathbb{F}_{2^n}$. Such curves are nonsingular and the commutativity condition (2.10) imposes the restrictions

$$\phi_k = \phi_{n-k}^{2^k}, \quad \psi_k = \psi_{n-k}^{2^k}, \quad k = 1, \dots, [(n-1)/2], \quad (2.15)$$

where $[]$ denotes the integer part. For even values of n , the additional requirement $\phi_{n/2} = \phi_{n/2}^{2^{n/2}}$ ($\psi_{n/2} = \psi_{n/2}^{2^{n/2}}$) should be satisfied.

The expansion coefficients are related to the curve equation in a very simple form. Indeed, by rewriting the equation $\beta = f(\alpha)$ as

$$\beta = f(\alpha) = f\left(\sum_{k=1}^n \text{tr}(\alpha \theta_k) \theta_k\right) = \sum_{k=1}^n \text{tr}(\alpha \theta_k) f(\theta_k) = \sum_{k=1}^n \sum_{i=0}^{n-1} \alpha^{2^i} \quad (2.16)$$

and comparing with (2.14), we immediately get that

$$\phi_i = \sum_{k=1}^n \theta_k^{2^i} f(\theta_k). \quad (2.17)$$

The degenerate curves have more involved structure [32]. The degeneration basically means that α and β do not take some values in the field, which is compensated by multiple appearance of other (admissible) “coordinates”, since the curve is nonsingular. Such curves are characterized by the degrees of degeneration r_α and r_β . This degree means the following: if (α_j, β_j) is a point of a degenerate curve, for each α_j there are 2^{n-r_β} values of β , such that the points (α_j, β_k) ($k = 1, \dots, 2^{n-r_\beta}$) belong to the same curve and, conversely, for each β_j there are 2^{n-r_α} values of α , such that the points (α_k, β_j) also belong to the same curve.

3. Operations with curves

We next show how the action of Clifford operations can be described in this finite grid. For a single qubit, the simplest of such operations are x - and z -rotations, which are represented by the unitaries $u_{x,z} = \exp(i\pi\sigma_{x,z}/4)$. They produce the transformations

$$x\text{-rotation} : \{\sigma_x \mapsto \sigma_x, \sigma_y \mapsto \sigma_z, \sigma_z \mapsto \sigma_y\}, \quad (3.1)$$

$$z\text{-rotation} : \{\sigma_x \mapsto \sigma_y, \sigma_y \mapsto \sigma_x, \sigma_z \mapsto \sigma_z\}.$$

Multiqubit rotations can be labeled by $\xi \in \mathbb{F}_{2^n}$, so that in the self-dual basis the expansion $\xi = \sum_i \xi_i \theta_i$ indicates on which qubit the rotation is performed: if $\xi_i = 1$, then the i -th qubit is rotated. The general form of a local x - and z -rotation is thus $U_{x,z}^\xi = \prod_{i=1}^n u_{x,z}^{\text{tr}(\xi \theta_i)}$. The indices of the monomials $Z_\alpha X_\beta$ transform then according to

$$\begin{aligned} x\text{-rotation} : \begin{cases} \alpha \mapsto \alpha, \\ \beta \mapsto \beta + \sum_k \xi_k \text{tr}(\alpha \theta_k) \theta_k, \end{cases} \\ z\text{-rotation} : \begin{cases} \alpha \mapsto \alpha + \sum_k \xi_k \text{tr}(\beta \theta_k) \theta_k, \\ \beta \mapsto \beta. \end{cases} \end{aligned} \quad (3.2)$$

With local transformations it is always possible to transform any degenerate curve into a nondegenerate one. Moreover, any curve can be cast into the form $\beta = f(\alpha)$, where $f(\alpha)$ is an invertible function, so that the curve equation can be also written as $\alpha = f^{-1}(\beta)$.

One- and two-qubit gates are crucial in quantum computation. As an archetypal example, we consider the nonlocal XOR _{ij} operator (applied to the i th and j th qubits) with the following action

$$\begin{aligned} \mathbb{1}^{(i)} \sigma_z^{(j)} &\mapsto \sigma_z^{(i)} \sigma_z^{(j)}, & \sigma_z^{(i)} \mathbb{1}^{(j)} &\mapsto \sigma_z^{(j)} \mathbb{1}^{(i)}, \\ \mathbb{1}^{(i)} \sigma_x^{(j)} &\mapsto \mathbb{1}^{(i)} \sigma_x^{(j)}, & \sigma_x^{(i)} \mathbb{1}^{(j)} &\mapsto \sigma_x^{(i)} \sigma_x^{(j)}. \end{aligned} \quad (3.3)$$

In our algebraic language this can be elegantly represented by

$$\alpha \mapsto \alpha + \theta_i \text{tr}(\alpha \theta_j), \quad \beta \mapsto \beta + \theta_j \text{tr}(\beta \theta_i), \quad (3.4)$$

so that

$$\text{XOR}_{ij} = \sum_{\lambda} |\lambda + \text{tr}(\lambda \theta_i) \theta_j\rangle \langle \lambda|. \quad (3.5)$$

Table 1. Irreducible polynomials and self-dual bases for the Galois fields \mathbb{F}_{2^n} used in this paper.

Field	Irreducible polynomial	Self-dual basis
\mathbb{F}_{2^2}	$x^2 + x + 1 = 0$	$\{\sigma, \sigma^2\}$
\mathbb{F}_{2^3}	$x^3 + x + 1 = 0$	$\{\sigma^3, \sigma^5, \sigma^6\}$
\mathbb{F}_{2^4}	$x^4 + x + 1 = 0$	$\{\sigma^3, \sigma^7, \sigma^{12}, \sigma^{13}\}$
\mathbb{F}_{2^5}	$x^5 + x^2 + 1 = 0$	$\{\sigma^3, \sigma^7, \sigma^{11}, \sigma^{15}, \sigma^{19}\}$

In particular, the curve $\beta = f(\alpha)$ is transformed into

$$\beta \mapsto f(\alpha) + \text{tr}(\alpha\theta_j)f(\theta_i) + \text{tr}[f(\alpha)\theta_i]\theta_j + \text{tr}(\alpha\theta_j)\text{tr}[f(\theta_i)\theta_i]\theta_j. \quad (3.6)$$

Equivalently, the expansion coefficients are $\phi_k \mapsto \phi_k + \theta_j^{2^k} f(\theta_i) + f^{2^k}(\theta_i)\theta_j$. For example, for two qubits, the curve $\beta = \alpha$ is transformed, under the action XOR_{12} , into $\beta = \sigma^2\alpha$; where σ is a primitive element of \mathbb{F}_{2^2} , solution of the irreducible polynomial $x^2 + x + 1 = 0$. For three qubits, the curve $\beta = \alpha$ becomes, also under the action XOR_{12} , $\beta = \sigma\alpha + \alpha^2 + \alpha^4$, where σ is the primitive element for \mathbb{F}_{2^3} , solution of $x^3 + x + 1 = 0$. In Table 1 we summarize the relevant information for the Galois fields used in this paper.

The SWAP operator exchanges the states of the i th and j th qubits; i.e.,

$$\text{SWAP}_{ij}|\dots, a_i, \dots, a_j, \dots\rangle = |\dots, a_j, \dots, a_i, \dots\rangle. \quad (3.7)$$

At the level of field elements, its action is

$$\alpha \mapsto \alpha + \varepsilon \text{tr}(\alpha\varepsilon), \quad \beta \mapsto \beta + \varepsilon \text{tr}(\beta\varepsilon), \quad (3.8)$$

where $\varepsilon = \theta_i + \theta_j$. Consequently, we can write

$$\text{SWAP}_{ij} = \sum_{\lambda} |\lambda + \varepsilon \text{tr}(\varepsilon\lambda)\rangle \langle \lambda|, \quad (3.9)$$

so it transforms the curve $\beta = f(\alpha)$ into

$$\beta \mapsto f(\alpha) + \text{tr}(\alpha\varepsilon)f(\varepsilon) + \text{tr}[f(\alpha)\varepsilon]\varepsilon + \text{tr}(\alpha\varepsilon)\text{tr}[f(\varepsilon)\varepsilon]\varepsilon. \quad (3.10)$$

The expansion coefficients of the transformed curve are $\phi_k + \varepsilon^{2^k} f(\varepsilon) + f^{2^k}(\varepsilon)\varepsilon$.

As our final nonlocal operation, we take squeezing, represented by

$$S_{\xi} = \sum_{\lambda} |\xi\lambda\rangle \langle \lambda|, \quad (3.11)$$

whose action on the basic monomials is $S_{\xi} Z_{\alpha} X_{\beta} S_{\xi}^{-1} = Z_{\xi\alpha} X_{\xi^{-1}\beta}$, so that, the curve is transformed into $\beta = \xi^{-2} f(\alpha)$, which, as expected, is just a global scaling

One can also define curve composition in such a way that given two curves $\beta = f(\alpha)$ and $\beta = g(\alpha)$, the function $\beta = f(g(\alpha))$ is also a curve iff $f(g(\alpha)) = g(f(\alpha))$.

The last remark concerning these curves is that they can always transformed into the horizontal axis $\beta = 0$. Indeed, one can always find an operator P_f such that

$$P_f Z_{\alpha} P_f^{-1} = \varphi_f(\alpha) Z_{\alpha} X_{f(\alpha)}, \quad (3.12)$$

where $\varphi_f(\alpha)$ is a phase factor. The operator P_f can be conveniently expanded in the the basis $\{|\tilde{\lambda}\rangle\}$ of eigenstates of X_{λ}

$$P_f = \sum_{\lambda} c_{\lambda}^{(f)} |\tilde{\lambda}\rangle \langle \tilde{\lambda}|. \quad (3.13)$$

The coefficients $c_\lambda^{(f)}$ can be determined by noting that

$$P_f Z_\alpha P_f^{-1} = \sum_\lambda c_{\lambda+\alpha}^{(f)} c_\lambda^{(f)*} |\widetilde{\lambda+\alpha}\rangle \langle \widetilde{\lambda}|, \quad (3.14)$$

while

$$Z_\alpha X_{f(\alpha)} = \sum_\lambda \chi[\lambda f(\alpha)] |\widetilde{\lambda+\alpha}\rangle \langle \widetilde{\lambda}|, \quad (3.15)$$

so that $c_\lambda^{(f)}$ satisfy the recurrence relation

$$c_{\lambda+\alpha}^{(f)} c_\lambda^{(f)*} = \phi_f(\alpha) \chi[\lambda f(\alpha)]. \quad (3.16)$$

The phase $\phi_f(\alpha)$ is evaluated by substituting $\lambda = \alpha$ in the above equation and taking $c_0^{(f)} = 1$, which finally leads to

$$c_\lambda^{(f)} c_{\lambda'}^{(f)} = \chi[\lambda' f(\lambda)] c_{\lambda+\lambda'}^{(f)}. \quad (3.17)$$

The solution of equation (3.17) is not unique, but choosing all $c_{\theta_i}^{(f)}$ for a basis to be positive, we arrive at the following explicit expression

$$c_\lambda^{(f)} = \left[\prod_{i=1}^n \sqrt{(-1)^{\ell_i M_{ii}^{(f)}}} \right] (-1)^{\ell^T \tilde{\mathbf{M}} \ell}. \quad (3.18)$$

Here $\ell = (\ell_1, \dots, \ell_n)$ is the vector of the expansion coefficients of $|\lambda\rangle$ in the self-dual basis, and the principal branch of the square root is chosen. In addition, $M_{ij}^{(f)} = \text{tr}[\sigma_i f(\sigma_j)]$ and

$$\tilde{\mathbf{M}} = \begin{cases} M_{ij}^{(f)} & j < i, \\ 0 & j > i. \end{cases} \quad (3.19)$$

All this means that given a curve we can immediately obtain the eigenstates $|\lambda; f\rangle$ of the set of commuting monomials corresponding to this curve: $|\lambda; f\rangle = P_f |\lambda\rangle$, with $|\lambda\rangle$ being the computational basis.

In a similar way, the curves $\alpha = g(\beta)$ can be transformed into the vertical axis $\alpha = 0$:

$$Z_{g(\beta)} X_\beta = \psi(\beta) Q_g X_\beta Q_g^{-1}, \quad (3.20)$$

where $\psi(\beta)$ is a phase. Again, one can show that Q_g can be expressed as

$$Q_g = \sum_{\kappa} c_\kappa^{(g)} |\widetilde{\kappa}\rangle \langle \widetilde{\kappa}|, \quad (3.21)$$

and the coefficients $c_\kappa^{(g)}$ satisfy (3.17).

4. Graph states and their phase space representation

Consider a graph $G = \{V, E\}$, which is a set of vertices $V = \{1, \dots, n\}$ connected in a specific way by a set of edges $E \subset V \times V$, specifying the neighborhood relation between vertices [37]. In the following, we only consider simple graphs, containing neither loops neither multiple edges. To each vertex i we attach a single qubit and each edge $\{i, j\}$ represents the interaction between the corresponding qubits. With the graph we associate a complete set of n commuting monomials

$$K_i = \sigma_z^{(i)} \prod_{j \in N_i} \sigma_x^{(j)}, \quad (4.1)$$

where N_i is the neighborhood for that vertex, $N_i = \{j \in V \mid \{i, j\} \in E\}$ (i.e., all the vertices connected to i). Please, note that we have interchanged the order of σ_z and σ_x with respect to the standard definition [12]: this corresponds to a global Clifford y -rotation, and will allow us to work with curves $\beta = f(\alpha)$ and not $\alpha = g(\beta)$.

The associated graph state $|G\rangle$ is just the unique common eigenvector of $\{K_i\}$ with all eigenvalues $+1$. The monomials $\{K_i\}$ are also the generators of the stabilizer of $|G\rangle$ [38], which is an Abelian subgroup of the local Pauli group \mathcal{P}_n [34]. Any stabilizer state is locally equivalent to some graph state [39] (more exactly, they are equivalent under local Clifford transformations).

The crucial observation for what follows is that the graph-state generators $\{K_i\}$ can be written precisely as $\{Z_\alpha X_{\beta=f(\alpha)}\}$, when α runs the self-dual basis $\{\theta_1, \dots, \theta_n\}$, if the function $\beta = f(\alpha)$ satisfies the extra condition

$$\text{tr}[\theta_i f(\theta_i)] = 0. \quad (4.2)$$

Note in passing that the x -rotation U_x^ξ , with

$$\xi = \sum_i \text{tr}[f(\theta_i)\theta_i] \theta_i, \quad (4.3)$$

reduces any curve $\beta = f(\alpha)$ to the form $\beta = f(\alpha) + \sum_i \text{tr}[f(\theta_i)\theta_i] \text{tr}(\alpha\theta_i) \theta_i$, which obviously satisfies the graph-state condition (4.2). For instance, in \mathbb{F}_{2^3} , the non-graph curve $\beta = \sigma^4 \alpha$ is transformed into the graph curve $\beta = \sigma^4 \alpha^2 + \sigma^2 \alpha^4$ by two x -rotations in qubits corresponding to elements of the self-dual basis σ^3 and σ^5 : $u_x^{\sigma^3} u_x^{\sigma^5}$.

It is indeed possible to find a generic form of the curves that fulfill (4.2). To this end, let us substitute the explicit expansion (2.14) into the condition (4.2):

$$\sum_{i=0}^{n-1} \text{tr}(\phi_i \theta_k^{2^i} \theta_k) = 0, \quad k = 1, \dots, n. \quad (4.4)$$

Since the expansion coefficients ϕ_i can be expressed in the form of equation (2.17), the condition $\text{tr}(\phi_0) = 0$ should be always satisfied for graph-state curves. When n is odd, and by using the property (2.15), we can reduce the sum (4.4) to the simpler form

$$\text{tr}(\phi_0^{2^{n-1}} \theta_k) = 0, \quad (4.5)$$

which shows that any curve for an odd number of qubits with $\phi_0 = 0$ corresponds to a graph.

When n is even, we have to take into account an additional condition on the coefficients, $\phi_{n/2} = \phi_{n/2}^{2^{n/2}}$, which leads to the restriction

$$\phi_0 = \sum_{i=1}^n \text{tr}(\phi_{n/2} \theta_i^{2^{n/2}} \theta_i) \theta_i^2. \quad (4.6)$$

A graph can be also characterized by its adjacency matrix Γ , which is a symmetric $n \times n$ matrix such that

$$\Gamma_{ij} = \begin{cases} 1, & \{i, j\} \in E \\ 0, & \{i, j\} \notin E. \end{cases} \quad (4.7)$$

The adjacency matrix for graphs associated to the curves fulfilling (4.2) has the form

$$\Gamma_{ij} = \text{tr}[\theta_i f(\theta_j)]. \quad (4.8)$$

For an arbitrary curve $\beta = f(\alpha)$, the result still holds once the diagonal part of Γ_{ij} in (4.8) is removed. Note, however, that the same adjacency matrix may correspond to different curves.

For instance, for three qubits the curves on \mathbb{F}_{2^3} $\beta = \sigma^4 \alpha$ and $\beta = \sigma^4 \alpha^2 + \sigma^2 \alpha^4$ have the same adjacency matrix: the second curve corresponds to a graph, while the first one does not.

Given the matrix Γ of a graph G , one can use (4.8) to immediately find that the corresponding $f(\theta_j)$ are

$$f(\theta_j) = \sum_{i=1}^n \text{tr}[\theta_i f(\theta_j)] \theta_i = \sum_{i=1}^n \Gamma_{ij} \theta_i. \quad (4.9)$$

Multiplying (4.9) by $\alpha_j = \text{tr}(\alpha \theta_j)$ and summing we obtain

$$\sum_{j=1}^n \alpha_j f(\theta_j) = \sum_{j=1}^n f(\alpha_j \theta_j) = f(\alpha) = \sum_{i,j=1}^n \alpha_j \Gamma_{ij} \theta_i, \quad (4.10)$$

so the curve corresponding to the matrix Γ is

$$\beta = \sum_{i,j=1}^n \text{tr}(\alpha \theta_j) \Gamma_{ij} \theta_i = \sum_{k=1}^n \sum_{i,j=1}^n \Gamma_{ij} \theta_j^{2^k} \theta_i \alpha^{2^k}, \quad (4.11)$$

or, equivalently, its expansion coefficients ϕ_k are

$$\phi_k = \sum_{i,j=1}^n \Gamma_{ij} \theta_i^{2^k} \theta_j. \quad (4.12)$$

Let us work out some examples. The simplest one is perhaps a linear chain of n qubits, with adjacency matrix

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & \ddots & \ddots & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad (4.13)$$

so the associated curve is

$$\beta = \sum_{i=2}^{n-1} \text{tr}(\alpha \theta_i) (\theta_{i+1} + \theta_{i-1}) + \text{tr}(\alpha \theta_1) \theta_2 + \text{tr}(\alpha \theta_n) \theta_{n-1}. \quad (4.14)$$

Another interesting example is a star graph, in which every single vertex is connected only to the center vertex (which we take as the n th qubit). Now the adjacency matrix is

$$\Gamma = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & \ddots & \ddots & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & \ddots & 1 & 0 \end{pmatrix}, \quad (4.15)$$

which leads to the very simple curve

$$\beta = \theta_n + \text{tr}(\alpha \theta_n). \quad (4.16)$$

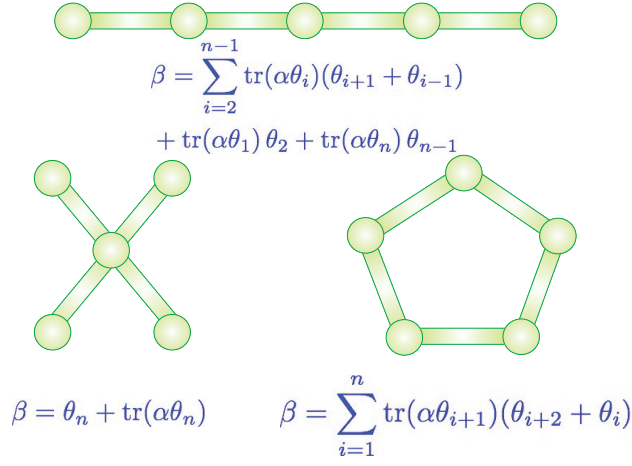


Figure 1. (Color online) Curves associated with different graph states. In all the cases, we have restricted ourselves to five qubits, although the curve equations are written for an arbitrary number n .

Finally, we consider a graph consisting of a closed polygon, with matrix

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 & 0 \\ 0 & \ddots & 0 & 1 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \quad (4.17)$$

and associated curve

$$\beta = \sum_{i=1}^n \text{tr}(\alpha \theta_{i+1})(\theta_{i+2} + \theta_i), \quad (4.18)$$

where the indices of the basis elements are taken mod n .

Please, observe carefully that for each of the examples worked out so far the following relation holds:

$$\beta(\theta_i) = \sum_{j \in N_i} \theta_j, \quad (4.19)$$

that is, the curve evaluated at each vertex appears as a sum of basis elements associated with all the connected vertices. In figure 1 we graphically summarize these relevant examples of the correspondence between phase-space curves and graphs.

Concerning the nonlocal operations XOR and SWAP, one can check that graph curves are transformed into graph curves.

Local Clifford operations on graph states can be easily explained in terms of the local complementation. For the graph $G = \{V, E\}$, the local complement of G at i is obtained by complementing the subgraph of G induced by the neighborhood N_i of i and leaving the rest of the graph unchanged. This is implemented by the local unitary

$$V_i = u_x^i \prod_{j \in N_i} u_z^j \quad (4.20)$$

The curve coefficients transform thus in a simple way

$$\phi_k \mapsto_{V_i} \phi_k + f^{2^k+1}(\theta_i) + \sum_{i=1}^n \text{tr}[f(\theta_i)\theta_i] \theta_i^{2^k+1}. \quad (4.21)$$

It is clear from that the local complement does not change the coefficient ϕ_0 , which reflects the fact that a graph is also transformed into a graph.

Finally, it is worth noting that for graph curves, the coefficients $c_\lambda^{(f)}$ in the expansion (3.17) simplify to

$$c_\lambda^{(f)} = (-1)^{\frac{1}{2}\ell'\Gamma\ell}, \quad (4.22)$$

where Γ is the adjacency matrix corresponding to the curve.

5. Factorization structure

One essential property of these curves is their factorization structure, which is related to the commutation condition between blocks of single-qubit operators.

Consider the commuting monomials labeled by points of a given curve. Each monomial is a direct product of n Pauli operators. Let us divide each monomial into two parts, so that the first part contains k Pauli operators, corresponding to the (i_1, \dots, i_k) qubits and the second part $n-k$ operators of the rest of the qubits. If any Pauli operator in the first block commutes with all the others in that block, we will say that the corresponding set of monomials is factorized at least into two subsets. Obviously, the second blocks would then also commute among themselves. Moreover, inside the first or second blocks some sub-blocks may exist that commute with corresponding sub-blocks, etc. Thus, we can represent any curve $\gamma \in \mathbb{F}_{2^n}$ in the following factorized form:

$$\gamma = \{m_1, m_2, \dots, m_N\}, \quad (5.1)$$

where $0 < m_1 \leq m_2 \leq \dots \leq m_N$ and $m_i \in \mathbb{N}$ is the number of particles in the i th block that cannot be factorized anymore. It is clear that $\{m_1, m_2, \dots, m_N\}$ is just a partition of the integer n , so the maximum number of terms is n , which corresponds to a completely factorized curve, $\gamma = \underbrace{\{1, 1, \dots, 1\}}_n$, and the minimum number of terms is one, corresponding to a completely nonfactorized curve $\gamma = \{n\}$.

The rays (straight lines passing through the origin) $\alpha = 0$, $\beta = 0$, and $\beta = \alpha$ are always completely factorized, while all the other are apparently completely nonfactorized for any dimension, except the special case of four qubits, where there are two locally equivalent rays with factorization (2,2).

It is easy to convince oneself that completely factorized curves can be written as

$$f(\alpha) = \sum_{i=1}^n \text{tr}(\alpha\theta_i) \xi_i \theta_i, \quad (5.2)$$

where ξ_i are the components of an arbitrary $\xi \in \mathbb{F}_{2^n}$ in the self-dual basis.

For bifactorized curves, whose adjacency matrix is just a $2k \times 2k$ permutation matrix, the corresponding form is

$$f(\alpha) = \sum_{n=1}^k [\text{tr}(\alpha\theta_{i_n}) \theta_{j_n} + \text{tr}(\alpha\theta_{j_n}) \theta_{i_n}], \quad (5.3)$$

where $\{i_1, j_1\}, \{i_2, j_2\}, \dots, \{i_k, j_k\}$ are ordered pairs of connected vertices. In this case, two relations are automatically satisfied: $f(f(\alpha)) = \alpha$ and $\text{tr}[f(\alpha)] = \text{tr}(\alpha)$.

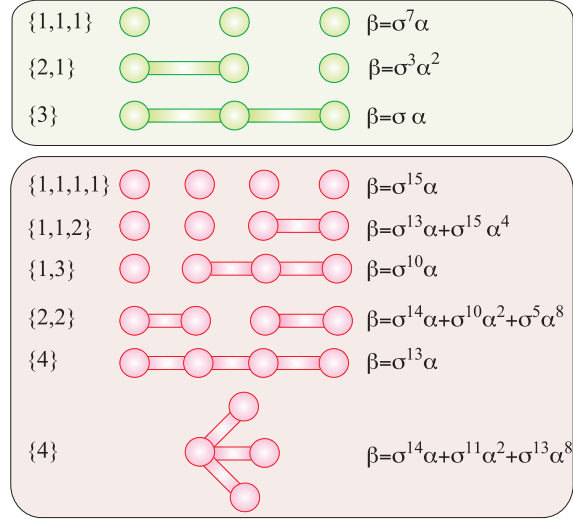


Figure 2. (Color online) Inequivalent curves for three (top) and four (bottom) qubits. In the left, we show the factorization structure; in the right, the curve in the simplest algebraic form and in the middle the locally equivalent graphs.

In general the problem of fully determining the factorization of a curve is quite difficult, and basically corresponds to the separation of an adjacency matrix in independent blocks by permutation of lines and columns. The number of such blocks can be determined by, e.g., counting the number of zero eigenvalues of the Laplacian matrix [40]. Nevertheless, some general considerations can be made. Let us evaluate a function $f(\alpha)$, corresponding to a graph, in an arbitrary element θ_i of a self-dual basis,

$$f(\theta_i) = \sum_{j=1}^n \text{tr}[f(\theta_i)\theta_j] \theta_j = \sum_{j=1}^n \Gamma_{ij} \theta_j, \quad (5.4)$$

which is the sum of elements of the basis corresponding to vertices (qubits) connected to the i th vertex. This means that this i th vertex is disconnected from the rest of the graph if $f(\theta_i) = 0$. Moreover, for each disconnected subgraph A_i , the function $f(\alpha)$ maps the subgraph onto itself $f(A_i) = A_i$ and thus

$$\text{tr} f \left(\sum_{j \in A_i} \theta_j \right) = 0 \quad (5.5)$$

for each separated block.

In figure 2 we list the locally inequivalent curves for three and four qubits, where they are expressed in the simplest algebraic form (not necessarily corresponding to graphs) and the equivalent graphs are plotted. This can be easily continued to higher number of qubits.

6. Some applications

6.1. Local and nonlocal transformations

The algebraic properties of the commuting stabilizers $\{Z_\alpha X_{f(\alpha)}\}$ allows us to scrutinize in a very simple way the action of both local and nonlocal transformations. Moreover, given

then relation between curves and adjacency matrices we can easily find how graphs are transformed, avoiding in this way involved procedures [39].

Let us start by considering the linear operations treated in section 3. It is clear from (3.3) that x -rotations (labelled by the parameter ξ , whose nonzero components indicate the qubits subjected to rotations) do not change the graph; they add only a diagonal element to the adjacency matrix: $\Gamma \mapsto \Gamma + \text{diag}(\xi_1, \dots, \xi_n)$.

On the other hand, z -rotations change the adjacency matrix in a nontrivial way, as hinted by (3.3). First, we find the corresponding curve for which we change the variable $\alpha \mapsto \mu = \alpha + \sum_{i=1}^n \xi_i \text{tr}[f(\alpha)\theta_i] \theta_i$. This requires the inversion of the system

$$\mu_i = \alpha_i + \xi_i \sum_{j=1}^n \alpha_j \Gamma_{ij}, \quad (6.1)$$

to be able to determine $\alpha = \alpha(\mu)$ and thus find the set $\{Z_\mu X_{g(\mu)}\}$, where $g = f(\alpha(\mu))$. Finally, the curve $\beta = g(\mu)$ can be converted into a graph-curve (if necessary) by some x -rotations, which immediately gives the new adjacency matrix.

As an example, let us analyze a four-qubit graph with adjacency matrix

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (6.2)$$

and find how it is transformed under z -rotations of the second and the fourth qubits. The curve corresponding to the matrix (6.2) is $\beta = \sigma^7 \alpha^2 + \sigma^{11} \alpha^8$. The rotation $\mathbb{1} \otimes u_z \otimes \mathbb{1} \otimes u_z$ transforms it into the curve $\beta = \sigma^{11} \alpha + \sigma^5 \alpha^2 + \sigma^{15} \alpha^4 + \sigma^{10} \alpha^8$. This means that

$$\Gamma \mapsto \Gamma' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \quad (6.3)$$

The final adjacency matrix is obtained from the above matrix just by removing the diagonal elements (which can be done just by applying the x -rotation $\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes u_x$). The resulting graph-curve is $\beta = \sigma^6 \alpha^2 + \sigma^{10} \alpha^4 + \sigma^3 \alpha^8$.

Next, we turn our attention to nonlocal operations. According to (3.6) the adjacency matrix associated with the graph-curve transformed by XOR_{ij} is

$$\Gamma_{ij} \mapsto \Gamma'_{kl} = \text{tr}[\theta_k f(\theta_l)] + \text{tr}[\theta_k f(\theta_i)] \delta_{lj} + \text{tr}[\theta_i f(\theta_l)] \delta_{kj}. \quad (6.4)$$

This can be rewritten as $\Gamma \mapsto \Gamma' = G \Gamma G^{-1}$, where $G_{kl} = \delta_{kl} + \delta_{kj} \delta_{li}$ is a triangular matrix with $\det G = 1$. For instance, the application of the gate XOR_{12} to the graph defined by the matrix (6.2) leads to

$$\Gamma' = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (6.5)$$

and the corresponding curve is $\beta = \sigma^{14} \alpha^2 + \sigma^7 \alpha^8$.

Finally, the squeezing operation (3.11) transforms the curve $\beta = f(\alpha)$ into $\beta = f(\mu \xi^{-1}) \xi^{-1}$ and, thus the adjacency matrix Γ corresponding to the initial graph into

$$\Gamma_{ij} \mapsto \text{tr}[\theta_i f(\theta_j \xi^{-1}) \xi^{-1}] = \text{tr} \left[\sum_{k=1}^n \text{tr}(\theta_i \xi^{-1} \theta_k) \theta_k f \left(\sum_{l=1}^n \text{tr}(\theta_j \xi^{-1} \theta_l) \theta_l \right) \right]$$

$$= \sum_{k,l=1}^n \text{tr}(\theta_i \xi^{-1} \theta_k) \text{tr}[\theta_k f(\theta_l)] \text{tr}(\theta_j \xi^{-1} \theta_l) = M_{ik} \Gamma_{kl} M_{lj}, \quad (6.6)$$

where $M_{ik} = \text{tr}(\theta_i \xi^{-1} \theta_k)$ is the matrix M for the ray $\beta = \xi^{-1} \alpha$.

For example, the graph corresponding to the adjacency matrix (6.2) is transformed under S_σ (where σ is the primitive element for \mathbb{F}_{2^4}) converts into the graph defined by

$$\Gamma' = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad (6.7)$$

and the corresponding curve is $\beta = \sigma^4 \alpha^2 + \sigma^2 \alpha^8$.

6.2. Computing reduced density matrices

As final illustration of the potential of our method, we consider the density matrix of the graph state corresponding to the curve $\beta = f(\alpha)$, which is given by

$$\rho = 2^{-n} \sum_{\alpha} Z_{\alpha} X_{f(\alpha)}. \quad (6.8)$$

Now assume that we trace over $n - k$ qubits. This leads to the following reduced density matrix

$$\tilde{\rho} = 2^{-k} \sum_{\alpha_1, \dots, \alpha_k \in \mathbb{Z}_2} Z_{\alpha_1 \theta_1 + \dots + \alpha_k \theta_k} X_{f(\alpha_1 \theta_1 + \dots + \alpha_k \theta_k)}. \quad (6.9)$$

With the machinery developed so far, this can be recast in the explicit form

$$\tilde{\rho} = 2^{-k} \sum_{\alpha_1, \dots, \alpha_k \in \mathbb{Z}_2} \prod_{i=1}^k \otimes \sigma_z^{\alpha_i} \sigma_x^{\text{tr}[f(\alpha_1 \sigma_1 + \dots + \alpha_k \sigma_k) \sigma_i]} = 2^{-k} \sum_{\alpha_1, \dots, \alpha_k \in \mathbb{Z}_2} \prod_{i=1}^k \otimes \sigma_z^{\alpha_i} \sigma_x^{\sum_{p=1}^k \alpha_p \Gamma_{pi}}, \quad (6.10)$$

where Γ is the adjacency matrix of the initial graph.

7. Conclusions

In summary, we have developed a phase-space picture of stabilizer states, i.e., a complete set of commuting monomials in the generalized Pauli group. This alternative description identifies these states with functions (which can be associated with specific nonsingular curves in discrete phase-space) over the finite field \mathbb{F}_2^n .

All the operations on graph states can be expressed in this language as transformations of the curves. Some of these transformations, such as the XOR gate or the discrete squeezing, are simpler to apply to the algebraic functions than to the graphs. On the contrary, the factorization is more involved at the algebraic level. We expect that the proposed method can be used for classification of graph states when the number of qubits is large and graphs do not provide any useful information. Work in this direction is in progress.

Acknowledgments

We would like to thank the two anonymous referees for their constructive comments. This work is partially supported by the Grant 106525 of CONACyT (Mexico), the Grants FIS2008-04356 and FIS2011-26786 of the Spanish DGI and the UCM-BSCH program (Grant GR-920992).

- [1]
- [2] Schlingemann D and Werner R F 2001 *Phys. Rev. A* **65** 012308
- [3] Raussendorf R and Briegel H J 2001 *Phys. Rev. Lett.* **86** 5188–5191
- [4] Raussendorf R, Browne D E and Briegel H J 2003 *Phys. Rev. A* **68** 022312
- [5] den Nest M V, Miyake A, Dür W and Briegel H J 2006 *Phys. Rev. Lett.* **97** 150504
- [6] Looi S Y, Yu L, Gheorghiu V and Griffiths R B 2008 *Phys. Rev. A* **78** 042303
- [7] Dür W, Calsamiglia J and Briegel H J 2005 *Phys. Rev. A* **71** 042336
- [8] Dür W, Aschauer H and Briegel H J 2003 *Phys. Rev. Lett.* **91** 107903
- [9] Han Y J, Raussendorf R and Duan L M 2007 *Phys. Rev. Lett.* **98** 150404
- [10] Scarani V, Acín A, Schenck E and Aspelmeyer M 2005 *Phys. Rev. A* **71** 042325
- [11] Gühne O and Cabello A 2008 *Phys. Rev. A* **77** 032108
- [12] Hein M, Dür W, Eisert J, Raussendorf R, den Nest M V and Briegel H J 2006 Entanglement in graph states and its applications *Quantum Computers, Algorithms and Chaos (International School of Physics "Enrico Fermi"* vol CLXII) ed Casati G, Shepelyansky D L, Zoller P and Benenti G Società Italiana di Fisica (IOS Press) pp 115–218
- [13] Briegel H J, Browne D E, Dür W, Raussendorf R and den Nest M V 2009 *Nat. Phys.* **5** 19–26
- [14] Lu C Y, X Q Zhou X Q, Gühne O, Gao W B, Zhang J, Yuan Z S, Goebel A, Yang T and Pan J W 2007 *Nat. Phys.* **3** 91–95
- [15] Vallone G, Pomarico E, De Martini F and Mataloni P 2008 *Phys. Rev. Lett.* **100** 160502
- [16] Gao W B, Xu P, Yao X C, Gühne O, Cabello A, Lu C Y, Peng C Z, Chen Z B and Pan J W 2010 *Phys. Rev. Lett.* **104** 020501
- [17] Gao W B, Lu C Y, Yao X C, Xu P, Gühne O, Goebel A, Chen Y A, Peng C Z, Chen Z B and Pan J W 2010 *Nat. Phys.* **6** 331–335
- [18] Hein M, Eisert J and Briegel H J 2004 *Phys. Rev. A* **69** 062311
- [19] Danielsen L E <http://www.iu.uib.no/~larsed/vncorbis/>
- [20] Cabello A, López-Tarrida A J, Moreno P and Portillo J R 2009 *Phys. Lett. A* **373** 2219–2225
- [21] Wootters W K 1987 *Ann. Phys.* **176** 1–21
- [22] Galetti D and de Toledo Piza A F R 1992 *Physica A* **186** 513–523
- [23] Gibbons K S, Hoffman M J and Wootters W K 2004 *Phys. Rev. A* **70** 062101
- [24] Wootters W K 2004 *IBM J. Res. Dev.* **48** 99–110
- [25] Vourdas A 2007 *J. Phys. A* **40** R285–R331
- [26] Lidl R and Niederreiter H 1986 *Introduction to Finite Fields and their Applications* (Cambridge: Cambridge University Press)
- [27] Galetti D and de Toledo Piza A F R 1988 *Physica A* **149** 267–282
- [28] Vourdas A 2005 *J. Phys. A* **38** 8453–8472
- [29] Cormick C, Galvão E F, Gottesman D, Paz J P and Pittenger A O 2006 *Phys. Rev. A* **73** 012301
- [30] Gross D 2007 *Appl. Phys. B* **86** 367–370
- [31] Björk G, Klimov A B and Sánchez-Soto L L 2008 *Prog. Opt.* **51** 469–516
- [32] Klimov A B, Romero J L, Björk G and Sánchez-Soto L L 2009 *Ann. Phys.* **324** 53–72
- [33] Klimov A B, Romero J L, Björk G and Sánchez-Soto L L 2007 *J. Phys. A* **40** 3987–3998
- [34] Chuang I and Nielsen M 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [35] Wootters W K and Fields B D 1989 *Ann. Phys.* **191** 363–381
- [36] Wootters W K 2006 *Found. Phys.* **36** 112–126
- [37] Diestel R 2000 *Graph Theory* (Heidelberg: Springer)
- [38] Gottesman D 1997 *Stabilizer codes and quantum error correction* Ph.D. thesis Caltech
- [39] Van den Nest M, Dehaene J and De Moor B 2004 *Phys. Rev. A* **69** 022316
- [40] Beineke L W and Wilson R J 2004 *Topics in Algebraic Graph Theory* (Cambridge: Cambridge University Press)